LINEARNI OPERATORI

$f: V_1 \to V_2$

$(\forall \alpha, \beta \in F)(\forall x, y \in V_1)\ f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

50. Neka je M_2 vekt. p. kvadratnih matrica nad F, i $f: M_2 \to M_2$ definisano sa: $(\forall X \in M_2)\ f(X) = X \cdot M$ pri čemu je $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ fiksirana matrica iz M_2

a) Dokazati da je f linearni operator.

b) Odrediti matricu operatora F u odnosu na bazu: $\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}$

a) $X, Y \in M_2,\ \alpha, \beta \in F$

$$f(\alpha X + \beta Y) = (\alpha X + \beta Y) \cdot M = (\alpha X) M + (\beta Y) M$$
$$= \alpha (XM) + \beta (YM) = \alpha f(X) + \beta f(Y)$$

$V_1;\ B_1 = \{e_1, \dots e_m\};\quad \dim V_1 = m$

$V_2;\ B_2 = \{f_1, \dots, f_m\};\quad \dim V_2 = m$

$(\forall x \in V_1)\quad f(x) = Ax$

$f(e_1) = a_{11} f_1 + a_{21} f_2 + \dots + a_{m1} f_m$

$f(e_2) = a_{12} f_1 + a_{22} f_2 + \dots + a_{m2} f_m$

$\vdots$

$f(e_n) = a_{1n} f_1 + a_{2n} f_2 + \dots + a_{mn} f_m$

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$f(e_1) = A \cdot e_1 = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = a_{11} f_1 + \dots + a_{m1} f_m$$

$$f(E_1) = E_1 \cdot M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} +$$

$$+ 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (a, b, 0, 0)$$

$$f(E_2) = E_2 \cdot M = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = (c, d, 0, 0)$$

$$f(E_3) = E_3 \cdot M = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = (0, 0, a, b)$$

$$f(E_4) = E_4 \cdot M = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} = (0, 0, c, d)$$

$$\Rightarrow A = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

51. – // – $f(X) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot X$ Rj. a) ✓

b) $A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$

52. $f: M_2 \to M_2$ nad poljem $\mathbb{R}$

$f(X) = A \cdot X \cdot B$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

a) lin. op. b) Naći matricu operatora

R:

$$A_f = \begin{bmatrix} ae & ag & be & bg \\ af & ah & bf & bh \\ ce & cg & de & dg \\ cf & ch & df & dh \end{bmatrix}$$

$\boxed{53.}$ $f: M_2 \to M_2$

$f(x) = AX + XB$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \quad A, B \in M_2$$

a) Da li je lin. op.

b) Naći matricu operatora f u odnosu na bazu:

$$B = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$ → nije stand. baza jer elementi nisu isti

$$f(E_1) = AE_1 + E_1 B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 0 & b \\ g & d+h \end{bmatrix}$$

$$= 0 \cdot E_4 + b \cdot E_3 + g \cdot E_2 + (d+h) \cdot E_1$$

$$= (d+h, g, b, 0)$$

$$A_f = \begin{pmatrix} d+h & f & c & 0 \\ g & d+e & 0 & c \\ b & 0 & a+h & f \\ 0 & b & g & a+e \end{pmatrix}$$

Standardna baza: $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$A_f = \begin{pmatrix} a+e & g & b & 0 \\ f & a+h & 0 & b \\ c & 0 & d+e & g \\ 0 & c & f & d+h \end{pmatrix}$$

centralna sim.

54. Dokazati da postoji jedinstven lin. operator 3D vektorskog prostora koji prevodi vektore a_1, a_2, a_3 u b_1, b_2, b_3, redom. Naći matricu tog operatora u istoj bazi u kojoj su date i koordinate tih vektora.

a) $a_1 = (2,3,5)$, $a_2 = (0,1,2)$, $a_3 = (1,0,0)$
$b_1 = (1,1,1)$, $b_2 = (1,1,-1)$, $b_3 = (2,1,2)$

b) $a_1 = (2,0,3)$, $a_2 = (4,1,5)$, $a_3 = (3,1,2)$
$b_1 = (1,2,-1)$, $b_2 = (4,5,-2)$, $b_3 = (1,-1,1)$

$$A = \frac{1}{3}\begin{pmatrix} -6 & 11 & 8 \\ -12 & 13 & 10 \\ 6 & -5 & -5 \end{pmatrix}$$

T: Lin. operator prostora U u prostor V je potpuno određen, ako su poznate slike elemenata baze prostora U.

a_1, a_2, a_3 lin. nez.?

$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$

$\vdots$

$\alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow a_1, a_2, a_3$ su lin. nezavisni

Pošto je prostor trodim. i a_1, a_2, a_3 su lin. nez. $\Rightarrow$ čine bazu prostora. $B = (a_1, a_2, a_3)$

$\Rightarrow \exists!$ linearni operator $f: V \to V$

$\{e_1, e_2, e_3\}$ baza

$f(a_1) = f(2,3,5) = f(2\cdot e_1 + 3e_2 + 5\cdot e_3)$

$\overset{f \text{ lin.op.}}{=} 2\cdot f(e_1) + 3f(e_2) + 5f(e_3) = (1,1,1)$

$= e_1 + e_2 + e_3$ (1)

$f(a_2) = f(0,1,2) = f(e_2) + 2f(e_3) = \overset{(1,1,-1)}{e_1 + e_2 - e_3}$ (2)

$f(a_3) = f(1,0,0) = f(e_1) = \overset{(2,1,2)}{2e_1 + e_2 + 2e_3}$ (3)

Rješavamo sis. po $f(e_i)$:

$$2f(e_1)+3f(e_2)+5f(e_3)=e_1+e_2+e_3$$
$$f(e_2)+2f(e_3)=e_1+e_2-e_3$$
$$f(e_1)=2e_1+e_2+2e_3$$

$$f(e_1)=2e_1+e_2+2e_3=(2,1,2)$$
$$2\cdot(2e_1+e_2+2e_3)+3f(e_2)+5f(e_3)=e_1+e_2+e_3$$
$$f(e_2)+2f(e_3)=e_1+e_2-e_3$$

$$3f(e_2)+5f(e_3)=-3e_1-e_2-3e_3$$
$$f(e_2)+2f(e_3)=e_1+e_2-e_3 \quad /\cdot(-3)$$

$$-f(e_3)=-6e_1-4e_2 \quad \Rightarrow \quad f(e_3)=(6,4,0)$$
$$f(e_2)=-11e_1-7e_2-e_3 \quad \Rightarrow \quad f(e_2)=(-11,-7,-1)$$

$$\Rightarrow \quad A_f=\begin{pmatrix} 2 & -11 & 6 \\ 1 & -7 & 4 \\ 2 & -1 & 0 \end{pmatrix}$$

55. Neka je V trodim. v.p. Ispitati koja od sljedećih preslikavanja $f: V \to V$ lin. operatori. U slučaju da su lin. operatori, naći njihove matrice u istoj bazi u kojoj su zadane koordinate vektora x i $f(x)$.

a) $f(x) = f(x_1, x_2, x_3) = (x_2 + x_3, 2x_1 + x_3, 3x_1 - x_2 + x_3)$

b) $f(x) = (x_1, x_2 + 1, x_3 + 2)$

c) $f(x) = (2x_1 + x_2, x_1 + x_3, x_3^2)$ nije

d) $f(x) = (x_1 - x_2 + x_3, x_3, x_2)$ jest $\quad A_f = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

a) $\forall \alpha, \beta \in F, \ \forall x, y \in V$

$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$

$= ((\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3), 2(\alpha x_1 + \beta y_1) + \alpha x_3 + \beta y_3, 3(\alpha x_1 + \beta y_1) -$

$- (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3))$

$= \dots = (\alpha(x_2 + x_3) + \beta(y_2 + y_3), \alpha(2x_1 + x_3) + \beta(2y_1 + y_3),$

$\alpha(3x_1 - x_2 + x_3) + \beta(3y_1 - y_2 + y_3)) =$

$= \alpha(x_2 + x_3, 2x_1 + x_3, 3x_1 - x_2 + x_3) + \beta(y_2 + y_3, 2y_1 + y_3, 3y_1 - y_2 + y_3)$

$= \alpha f(x) + \beta f(y)$

1. način:

$B = \{e_1, e_2, e_3\}$

$f(e_1) = f(1,0,0) = (0,2,3)$

$f(e_2) = f(0,1,0) = (1,0,-1) \quad \Rightarrow A_f = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}$

$f(e_3) = f(0,0,1) = (1,1,1)$

2. način (u ovom slučaju možemo direktno odrediti mat. operatora)

A_f; $f(x) = A_f \cdot x$

da bi bilo $f(x)$ treba:

$$\begin{pmatrix} x_2 + x_3 \\ 2x_1 + x_3 \\ 3x_1 - x_2 + x_3 \end{pmatrix} = A_f \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow A_f = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}$$

b) $f(\alpha x+\beta y) \overset{?}{=} \alpha f(x) + \beta f(y)$

$$(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2 + 1, \alpha x_3 + \beta y_3 + 2) = (\alpha x_1, \alpha x_2 + \alpha, \alpha x_3 + 2\alpha) +$$
$$+ (\beta y_1, \beta y_2 + \beta, \beta y_3 + 2\beta)$$
$$\neq (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2 + \alpha + \beta, \alpha x_3 + \beta y_3 + 2(\alpha+\beta))$$

nije lin. operator!

56. Neka je P_3 v.p. polinoma stepena ≤ 3 nad poljem $\mathbb{R}$, i $f: P_3 \to P_3$ definisano sa:

$$f(P(t)) = \int_0^1 P(t+\lambda)\,d\lambda \qquad \forall P \in P_3$$

a) Dokazati da je f lin. operator.
b) Naći matricu operatora f u odnosu na bazu $\{1, t, t^2, t^3\}$

a) $(\forall \alpha, \beta \in \mathbb{R})\ (\forall P, Q \in P_3)$

$$f(\alpha P + \beta Q) = \int_0^1 (\alpha P + \beta Q)(t+\lambda)\,d\lambda = \int_0^1 ((\alpha P)(t+\lambda) + (\beta Q)(t+\lambda))\,d\lambda =$$
$$= \int_0^1 (\alpha \cdot P(t+\lambda) + \beta Q(t+\lambda))\,d\lambda$$
$$= \alpha \cdot \int_0^1 P(t+\lambda)\,d\lambda + \beta \int_0^1 Q(t+\lambda)\,d\lambda = \alpha f(P) + \beta f(Q)$$

b) $f(1) = \int_0^1 1 \cdot d\lambda = 1 = (1, 0, 0, 0) = (1 \cdot 1, 0 \cdot t, 0 \cdot t^2, 0 \cdot t^3)$

$P(t) = t$

$$f(t) = \int_0^1 (t+\lambda)\,d\lambda = \left(t\lambda + \frac{\lambda^2}{2}\right)\Big|_0^1 = t + \frac{1}{2} = \left(\frac{1}{2}, 1, 0, 0\right)$$

$$f(t^2) = \int_0^1 (t+\lambda)^2\,d\lambda = \int_0^1 (t^2 + 2\lambda t + \lambda^2)\,d\lambda = \left(t^2\lambda + 2\frac{\lambda^2}{2}t + \frac{\lambda^3}{3}\right)\Big|_0^1$$
$$= t^2 + t + \frac{1}{3} = \left(\frac{1}{3}, 1, 1, 0\right)$$

$$f(t^3) = \int_0^1 (t+r)^3 dr = \int_0^1 (t^3 + 3t^2 r + 3t r^2 + r^3) dr = \left(t^3 r + 3t^2 \frac{r^2}{2} + 3t \frac{r^3}{3} + \frac{r^4}{4}\right)\Big|_0^1$$

$$= t^3 + \frac{3t^2}{2} + t + \frac{1}{4} = \left(\frac{1}{4}, 1, \frac{3}{2}, 1\right)$$

$$\Rightarrow A_f = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

57. Neka je V skup svih realnih f-ja oblika $f(x) = e^{ax} \cdot P(x)$, gdje je $a \in \mathbb{R}$, $a \neq 0$, a $P(x)$ je realan polinom stepena ≤ 3. Dokazati da je operator definisan sa $D(f) = f'$ linearan o. prostora V i naći matricu tog operatora u odnosu na bazu $B = \{e^{ax}, xe^{ax}, x^2 e^{ax}, x^3 e^{ax}\}$

a) $D(\alpha f + \beta g) = \ldots = \alpha D(f) + \beta D(g)$

b) $$mat_B D = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & a & 3 \\ 0 & 0 & 0 & a \end{pmatrix}$$

58. Neka je $P_n[x]$ v.p. pol. stepena $\leq n$ nad $\mathbb{R}$ i A preslikavanje definisano sa:

$$A(P(x)) = (cx+d)^n \cdot P\left(\frac{ax+b}{cx+d}\right), \quad a,b,c,d \in \mathbb{R}, \quad ad-bc \neq 0$$

a) Dokazati da je A lin. op. prostora $P_n[x]$

b) Naći mat. operatora u odnosu na bazu $\{1, x, x^2, \ldots, x^n\}$ za $a=d=0$, $b=c=1$

$$A(\alpha P + \beta Q) = (cx+d)^n (\alpha P + \beta Q)\left(\frac{ax+b}{cx+d}\right)$$

$$= \ldots = (cx+d)^n \left(\alpha \cdot P\left(\frac{ax+b}{cx+d}\right) + \beta Q\left(\frac{ax+b}{cx+d}\right)\right)$$

$$= \alpha (cx+d)^n \cdot P\left(\frac{ax+b}{cx+d}\right) + \beta (cx+d)^n Q\left(\frac{ax+b}{cx+d}\right) = \alpha A(P) + \beta A(Q)$$

b) $a=d=0,\ b=c=1$.

$$A(P) = x^n \cdot P\left(\frac{1}{x}\right)$$

$$A(1) = x^n \cdot 1 = x^n = (0,0,\dots,0,1)$$

$$A(x) = x^n \cdot \frac{1}{x} = x^{n-1} = (0,\dots,0,1,0)$$

$\vdots$

$$A(x^n) = x^n \cdot \frac{1}{x^n} = 1 = (1,0,\dots,0)$$

$P = \frac{1}{x}$

$P_{(x^n)} = \frac{1}{x^n}$

$$mat_B A = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

$\boxed{59}$ Linearan operator u prostoru $P_2[x]$ realnih pol. stepena ≤ 2, ima matricu u odnosu na bazu $\{1, x, x^2\}$, ovakvu: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Naći matricu tog operatora u odnosu na bazu: $\{\overset{B_1}{3x^2+x+1}, \overset{B_2}{x^2+3x+2}, \overset{B_3}{2x^2+x+3}\}$

treba naći sliku operatora i izraziti preko vekt.

Uputa: Naći sliku: $f(3x^2+x+1) = \dots = \alpha_1(3x^2+x+1) + \alpha_2 \cdot (x^2+3x+$ $+ \alpha_3(2x^2+x+3)$ i ostale ... , 3 sis. za svaku kol.

$$\Rightarrow mat_{B_1} f = \begin{pmatrix} \alpha_1 & \cdot & \cdot \\ \alpha_2 & \cdot & \cdot \\ \alpha_3 & \cdot & \cdot \end{pmatrix}$$

Rj. $\frac{1}{17}\begin{pmatrix} -7 & 12 & 12 \\ 2 & 16 & -1 \\ 18 & -9 & 8 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$$f(3x^2+x+1) = \alpha_1(3x^2+x+1)+\alpha_2(x^2+3x+2)+\alpha_3(2x^2+x+3)$$

$$(3\alpha_1+\alpha_2+2\alpha_3)x^2+(\alpha_1+3\alpha_2+\alpha_3)x+(\alpha_1+2\alpha_2+3\alpha_3) = 3\cdot 1+x+x^2$$

$$3\alpha_1+\alpha_2+2\alpha_3=1$$

$$\alpha_1+3\alpha_2+\alpha_3=1$$

$$\alpha_1+2\alpha_2+3\alpha_3=3$$

$$D=\begin{vmatrix}3&1&2\\1&3&1\\1&2&3\end{vmatrix} /\cdot(-1)/\cdot(-3) = \begin{vmatrix}0&-8&-1\\1&3&1\\0&-1&2\end{vmatrix} = -1\cdot(-16-1)=17$$

$$D_1=\begin{vmatrix}1&1&2\\1&3&1\\3&2&3\end{vmatrix} /\cdot(-1)/\cdot(-3) \begin{vmatrix}1&1&2\\0&2&-1\\0&-1&-3\end{vmatrix} = -6+1=-7 \qquad \alpha_1=\frac{D_1}{D}=\frac{-7}{17}$$

$$D_2=\begin{vmatrix}3&1&2\\1&1&1\\1&3&3\end{vmatrix} /\cdot(-1)/\cdot(-3) \begin{vmatrix}3&1&2\\-2&0&-1\\-8&0&-3\end{vmatrix} = -1\cdot(6-8)=2 \qquad \alpha_2=\frac{D_2}{D}=\frac{2}{17}$$

$$D_3=\begin{vmatrix}3&1&1\\1&3&1\\1&2&3\end{vmatrix} /\cdot(-1)/\cdot(-3) \begin{vmatrix}3&1&1\\-2&2&0\\-8&-1&0\end{vmatrix} = 1\cdot(2+16)=18 \qquad \alpha_3=\frac{D_3}{D}=\frac{18}{17}$$

$f(3x^2+x+1) = 3\cdot f(x^2)+f(x)+f(1)$ zato što je lin. operator

$$= 3\cdot 1+x+x^2 = \alpha_1(3x^2+x+1)+\alpha_2(x^2+3x+2)+\alpha_3(2x^2+x+3)$$

$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ mat. u odnosu na $\{1, x, x^2\}$

$$\Rightarrow f(1)=0\cdot 1+0\cdot x+1\cdot x^2$$

$$f(x)=0\cdot 1+1\cdot x+0\cdot x^2$$

$$f(x^2)=1\cdot 1+0\cdot x+0\cdot x^2$$

$$f(x^2+3x+2) = f(x^2)+3f(x)+2f(1) = 1-3x+2x^2 =$$

$$= 1+3x+2x^2 = \alpha_1(3x^2+x+1)+\alpha_2(x^2+3x+2)+\alpha_3(2x^2+x+3)$$

$$3\alpha_1+\alpha_2+2\alpha_3 = 2 \quad /\cdot(-2)$$

$$\alpha_1+3\alpha_2+\alpha_3 = 3$$

$$\alpha_1+2\alpha_2+3\alpha_3 = 1 \quad /\cdot(-1)$$

$$3\alpha_1+\alpha_2+2\alpha_3 = 2$$

$$\alpha_2-2\alpha_3 = 2 \quad /\cdot(-1)$$

$$-5\alpha_1 - \alpha_3 = -3$$

$$\alpha_2 = 2+2\alpha_3 = 2-\frac{18}{17} = \frac{16}{17}$$

$$3\alpha_1 + 4\alpha_3 = 0$$

$$-5\alpha_1 - \alpha_3 = -3 \quad /\cdot 4$$

$$-17\alpha_1 = -12 \qquad \boxed{\alpha_1 = \frac{12}{17}}$$

$$\alpha_3 = 3-5\cdot\frac{12}{17} = \frac{-9}{17}$$

$$f(2x^2+x+3) = 2f(x^2)+f(x)+3f(1)$$

$$2+x+3x^2 = \alpha_1(3x^2+x+1)+\alpha_2(x^2+3x+2)+\alpha_3(2x^2+x+3)$$

$$3\alpha_1+\alpha_2+2\alpha_3 = 3$$

$$\alpha_1+3\alpha_2+\alpha_3 = 1$$

$$\alpha_1+2\alpha_2+3\alpha_3 = 2$$

$$D = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} \\ /\cdot(-3) \\ /\cdot(-1) \end{matrix} = \begin{vmatrix} 0 & -8 & -1 \\ 1 & 3 & 1 \\ 0 & -1 & 2 \end{vmatrix} = -(-16-1) = 17$$

$$\alpha_1 = \frac{12}{17}$$

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -1 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & -2 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 10+2 = 12$$

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix} = -1\cdot(3\cdot 0+1\cdot 1) = -1 \qquad \alpha_2 = \frac{-1}{17}$$

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1\cdot(9-1) = 8 \qquad \alpha_3 = \frac{8}{17}$$

60. Linearno preslikavanje $A: V \to W$ u odnosu na baze $\{e_1, e_2, e_3\}$ prostora V i $\{f_1, f_2\}$ prostora W ima matricu $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$. Naći matricu preslikavanja A, u odnosu na baze $\{e_1, e_1+e_2, e_1+e_2+e_3\}$ prostora V i $\{f_1, f_1+f_2\}$ prostora W.

$$\left.\begin{array}{l} A(e_1) = \alpha_1 \cdot f_1 + \alpha_2 (f_1 + f_2) \\ A(e_1) = 0 \cdot f_1 + 3 \cdot f_2 \end{array}\right\} \Rightarrow \alpha_1 + \alpha_2 = 0$$

$$\alpha_2 = 3 \Rightarrow \alpha_1 = -3$$

$$\left.\begin{array}{l} A(e_1+e_2) = \alpha_1 \cdot f_1 + \alpha_2 (f_1 + f_2) \\ A(e_1+e_2) = A(e_1) + A(e_2) \\ \quad = 0 \cdot f_1 + 3 \cdot f_2 + 1 \cdot f_1 + 4 \cdot f_2 \end{array}\right\} \Rightarrow \alpha_1 + \alpha_2 = 1$$

$$\alpha_2 = 7 \Rightarrow \alpha_1 = -6$$

$$\left.\begin{array}{l} A(e_1+e_2+e_3) = \alpha_1 f_1 + \alpha_2 (f_1 + f_2) \\ A(e_1+e_2+e_3) = A(e_1) + A(e_2) + A(e_3) \\ \quad = 0 \cdot f_1 + 1 \cdot f_1 + 2 \cdot f_1 + 3 \cdot f_2 + 4 \cdot f_2 + 5 \cdot f_2 \end{array}\right\} \Rightarrow \alpha_1 + \alpha_2 = 3$$

$$\alpha_2 = 12 \Rightarrow \alpha_1 = -9$$

$$\begin{pmatrix} -3 & -6 & -9 \\ 3 & 7 & 12 \end{pmatrix}$$

61. Matrica linearnog prostora $\mathbb{R}^3$ u bazi $\{\overset{e_1}{(8,-6,7)}, \overset{e_2}{(-16,7,-13)}, \overset{e_3}{(9,-3,7)}\}$ je $\begin{pmatrix} 1 & -18 & 15 \\ -1 & -22 & 20 \\ 1 & -25 & 22 \end{pmatrix}$. Naći matricu istog operatora, ali u bazi $\{\overset{x_1}{(1,-2,1)}, \overset{x_2}{(3,-1,2)}, \overset{x_3}{(2,1,2)}\}$.

$f(e_3) = 15 e_1 + 20 e_2 + 22 e_3$

$f(e_1) = 1 \cdot e_1 - 1 \cdot e_2 + 1 \cdot e_3$

1. izraziti preko e_1, e_2, e_3
$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$f(1,-2,1) = \alpha_1 \cdot (1,-2,1) + \alpha_2 (3,-1,2) + \alpha_3 (2,1,2)$

3 sistema
i matrica kad se
bude radila još 3 sis.

$$\begin{array}{l} \alpha_1 + 3\alpha_2 + 2\alpha_3 = 1 \\ -2\alpha_1 - \alpha_2 + \alpha_3 = -2 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 = 1 \end{array} \qquad \alpha_1 = 1 - 3\alpha_2 - 2\alpha_3$$

S: $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

Neka je $\mathcal{A}: V \to W$, linearni operator:

$\mathrm{Ker}\,\mathcal{A} = \{x \in V \mid \mathcal{A}x = o\}$, $o \in W$ – jezgro operatora
O_W; $\mathrm{Ker}\,\mathcal{A}$ je v.p.

$\mathrm{Im}\,\mathcal{A} = \{\mathcal{A}x \mid x \in V\}$, slika operatora,

$\mathrm{def}\,\mathcal{A} = \dim(\mathrm{Ker}\,\mathcal{A})$ defekt operatora,

$\mathrm{rang}\,\mathcal{A} = \dim(\mathrm{Im}\,\mathcal{A})$ rang operatora.

T: $\mathcal{A}: V \to V$ onda važi sljedeće:

$\dim V = \mathrm{rang}\,\mathcal{A} + \mathrm{def}\,\mathcal{A}$.

62. Neka je lin. operator $\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^4$ definisan sa

$\mathcal{A}(1,0,0) = (0,1,0,2)$

$\mathcal{A}(0,1,0) = (0,1,1,0)$

$\mathcal{A}(0,0,1) = (0,1,-1,4)$

Naći $\mathrm{Ker}\,\mathcal{A}$, $\mathrm{def}\,\mathcal{A}$, $\mathrm{Im}\,\mathcal{A}$ i $\mathrm{rang}\,\mathcal{A}$.

$\mathrm{Ker}\,\mathcal{A} = \{x \in \mathbb{R}^3 \mid \mathcal{A}x = \underbrace{0}_{(0,0,0,0)}\}$

$\mathcal{A}x = \mathcal{A}(x_1, x_2, x_3) = \mathcal{A}(x_1 \cdot (1,0,0) + x_2(0,1,0) + x_3(0,0,1))$

pošto je linearan:

$= x_1 \mathcal{A}(1,0,0) + x_2 \mathcal{A}(0,1,0) + x_3 \mathcal{A}(0,0,1)$

$= x_1(0,1,0,2) + x_2(0,1,1,0) + x_3(0,1,-1,4)$

$= \ldots = \underbrace{(0, x_1+x_2+x_3, x_2-x_3, 2x_1+4x_3)}_{\mathcal{A}x} = (0,0,0,0)$

$x_1 + x_2 + x_3 = 0$ $\quad x_1 = -2x_3$

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

$2x_1 + 4x_3 = 0$ $\quad \Rightarrow -4x_3 + 4x_3 = 0 \Rightarrow 0 = 0$

$(-2x_3, x_3, x_3) = x_3(-2,1,1)$

$\Rightarrow \mathrm{Ker}\,\mathcal{A} = \mathcal{L}\{(-2,1,1)\}$

jezgro je linearni omotač jednog vektora

$\operatorname{def} \mathcal{A} = \dim(\operatorname{Ker} \mathcal{A}) = 1$

$\operatorname{Im} \mathcal{A} = \{\mathcal{A}x \mid x \in \mathbb{R}^3\}$

$\cancel{= (0, x_1 + x_2 + x_3, x_2 - x_3, 2x_1 + 4x_3)}$

$\operatorname{Im} \mathcal{A} = x_1(0,1,0,2) + x_2(0,1,1,0) + x_3(0,1,-1,4)$

$= \mathcal{L}\{\underbrace{(0,1,0,2),(0,1,1,0),(0,1,-1,4)}\} \overset{*}{=}$

treba provjeriti da li su lin. nezavisni

$\alpha_1(0,1,0,2) + \alpha_2(0,1,1,0) + \alpha_3(0,1,-1,4) = 0$

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{pmatrix} /(-1) \sim \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rang $A = 2$

$\overset{*}{=} \mathcal{L}\{\underbrace{(0,1,0,2),(0,0,1,-2)}\}$

$\operatorname{Im} \mathcal{A}$ → lin. omotač svih lin. nezavisnih vek.

$\operatorname{rang} \mathcal{A} = \dim(\operatorname{Im} \mathcal{A}) = 2$

Provjera teoreme: $\dim \mathbb{R}^3 = \operatorname{def} \mathcal{A} + \operatorname{rang} \mathcal{A}$

$3 = 1 + 2$ ✓

63. Neka je $\{b_1, b_2, b_3\}$ baza trodimenzionalnog v.p. V i neka je $\mathcal{A}: V \to V^3$ linearan operator definisan sa: $\mathcal{A}b_1 = b_1 + b_2 - b_3$

$\mathcal{A}(b_2) = b_1 - b_2 + b_3$

$\mathcal{A}(b_3) = b_1 - 3b_2 + 3b_3$

Naći $\operatorname{def} \mathcal{A}$ i $\operatorname{rang} \mathcal{A}$.

$\operatorname{def} \mathcal{A} = 1$

$\operatorname{rang} \mathcal{A} = 2$

→ prvo jezgro i sliku

6h. $\mathcal{A} \in Hom(\mathbb{R}^4, \mathbb{R}^4)$ tj. $\mathcal{A}$ je lin. operator $\mathcal{A}: \mathbb{R}^4 \to \mathbb{R}^4$ definisan sa $\mathcal{A}(x,y,z,t) = (x+2y+3z, 2x+3y+t, x-3z+2t, -y+t)$. Odrediti jednu bazu prostora $Ker\,\mathcal{A}$, i $Im\,\mathcal{A}$.

$Ker\,\mathcal{A} = \{x \in \mathbb{R}^4 \mid \mathcal{A}x = 0\}$

$(0,0,0,0)$ - iz prostora slike

$= \{(x,y,z,t) \mid (x+2y+3z, 2x+3y+t, x-3z+2t, -y+t) = (0,0,0,0)\}$

$$\begin{aligned} x+2y+3z &= 0 \\ 2x+3y+t &= 0 \\ x-3z+2t &= 0 \\ -y+t &= 0 \end{aligned}$$

$$\begin{aligned} x+2y+3z &= 0 \\ 2x+3y+t &= 0 \quad /(-1) \\ 2x+2y+2t &= 0 \\ -y+t &= 0 \end{aligned}$$

$$\begin{aligned} x+2y+3z &= 0 \\ 2x+3y+t &= 0 \\ -y+t &= 0 \\ -y+t &= 0 \end{aligned} \Big\} \; y = t$$

$-2t+2t+3z = 0 \Rightarrow z = 0$

$2x+4t = 0 \qquad x = -2t$

$\Rightarrow (-2t, t, 0, t) = t(-2,1,0,1)$

$\Rightarrow Ker\,\mathcal{A} = \mathcal{L}\{(-2,1,0,1)\}$

jedan jedini vektor je baza $Ker\,\mathcal{A}$

$\Rightarrow def\,\mathcal{A} = 1$

$Im\,\mathcal{A} = \{\mathcal{A}x \mid x \in \mathbb{R}^4\} =$

$= \{(x+2y+3z, 2x+3y+t, x-3z+2t, -y+t) \mid (x,y,z,t) \in \mathbb{R}^4\}$

$= \{x(1,2,1,0) + y(2,3,0,-1) + z(3,0,-3,0) + t(0,1,2,1) \mid (x,y,z,t)\ldots$

$= \mathcal{L}\{(1,2,1,0), (2,3,0,-1), (3,0,-3,0), (0,1,2,1)\} \neq$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & -1 \\ 3 & 0 & -3 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{matrix} /\cdot(-2) & /\cdot(-3) \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & -6 & -6 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} /\cdot 6 \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -18 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{*}{=} \mathcal{L}\{(1,2,1,0), \underbrace{(0,-1,-2,-1)}_{\cdot(-1)}, \underbrace{(0,0,-18,-6)}_{:(-6)}\}$$

$$= \mathcal{L}\{\underbrace{(1,2,1,0),(0,1,2,1),(0,0,3,1)}_{\text{čine bazu } B \text{ prostora } Im\mathcal{A}}\}$$

$\Rightarrow$ rang $\mathcal{A} = 3$

ZADAĆA:

63. $\{b_1, b_2, b_3\}$ $\mathcal{A}: V \to V$ linearan operator definisan sa:

$\mathcal{A}b_1 = b_1 + b_2 - b_3$

$\mathcal{A}b_2 = b_1 - b_2 + b_3$

$\mathcal{A}b_3 = b_1 - 3b_2 + 3b_3$

def $\mathcal{A} = ?$ rang $\mathcal{A} = ?$

def $\mathcal{A} = \dim(Ker\mathcal{A})$; rang $\mathcal{A} = \dim(Im(\mathcal{A}))$

$Ker\mathcal{A} = \{x \in V : \mathcal{A}x = 0_V\} = \{(b_1,b_2,b_3) \mid (\mathcal{A}b_1; \mathcal{A}b_2; \mathcal{A}b_3) = (0,0,0)\}$

$$\left.\begin{matrix} b_1 + b_2 - b_3 = 0 \\ b_1 - b_2 + b_3 = 0 \end{matrix}\right\} + \quad 2b_1 = 0 \Rightarrow b_1 = 0$$

$$\underline{b_1 - 3b_2 + 3b_3 = 0} \qquad -3b_2 = -3b_3 \Rightarrow b_2 = b_3$$

$$(0, b_2, b_2) = b_2(0,1,1)$$

$Ker\mathcal{A} = \mathcal{L}\{(0,1,1)\} \Rightarrow$ def $\mathcal{A} = 1$

$Im\mathcal{A} = \{y \in V : (\exists x \in V)\ y = \mathcal{A}x\} = (\mathcal{A}b_1, \mathcal{A}b_2, \mathcal{A}b_3)$

$= \{(b_1 + b_2 - b_3;\ b_1 - b_2 + b_3;\ b_1 - 3b_2 + 3b_3)\}$

$= \{b_1(1,1,1) + b_2(1,-1,-3) + b_3(-1,1,3)\}$

$= \mathcal{L}\{(1,1,1), (1,-1,-3), (-1,1,3)\} \overset{*}{=}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ -1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & -2 \\ 1 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rang } A = 2$$

$\stackrel{*}{=} \mathcal{L}\{(2,0,-2),(1,-1,-3)\}$ $\quad$ rang A + def $A = 3$

$2 + 1 = 3$

61. $\{(8,-6,7),(-16,7,-13),(9,-3,7)\}$ baza $\mathbb{R}^3$

$$\begin{pmatrix} 1 & -18 & 15 \\ -1 & -22 & 20 \\ 1 & -25 & 22 \end{pmatrix}$$

Naći mat. istog operatora u bazi $\{(1,-2,1),(3,-1,2),(2,1,2)\}$

$f(1,-2,1) = \alpha_1(1,-2,1) + \alpha_2(3,-1,2) + \alpha_3(2,1,2)$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 1 \quad / \cdot (-1)$ $\quad$ 33

$-2\alpha_1 - \alpha_2 + \alpha_3 = -2$ $\quad$ -16

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ $\quad$ 27

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 1$

$-2\alpha_1 - \alpha_2 + \alpha_3 = -2$

$-\alpha_2 = 0 \Rightarrow \alpha_2 = 0$

$\alpha_1 + 2\alpha_3 = 1 \quad / \cdot 2$

$-2\alpha_1 + \alpha_3 = -2$

$5\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \Rightarrow \alpha_1 = 1$

$\alpha_2 = -33 + 27 \Rightarrow \boxed{\alpha_2 = 6}$

$\alpha_1 + 2\alpha_3 = 15 \quad / \cdot 2$

$-2\alpha_1 + \alpha_3 = -10$

$5\alpha_3 = 20$

$\alpha_3 = 4$

$\alpha_1 = 7$

$f(3,-1,2) = \alpha_1(1,-2,1) + \alpha_2(3,-1,2) + \alpha_3(2,1,2)$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 3 \quad / \cdot (-1)$ $\quad$ -17

$-2\alpha_1 - \alpha_2 + \alpha_3 = -1$ $\quad$ 28

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 2$ $\quad$ -15

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 3$

$-2\alpha_1 - \alpha_2 + \alpha_3 = -1$

$-\alpha_2 = -1 \Rightarrow \alpha_2 = 1$

$-\alpha_2 = 2 \Rightarrow \boxed{\alpha_2 = -2}$

$\alpha_1 + 2\alpha_3 = -11 \quad / \cdot 2$

$-2\alpha_1 + \alpha_3 = 27$

$5\alpha_3 = 5$

$\alpha_3 = 1$

$\alpha_1 = -13$

$\alpha_1 + \cancel{3} + 2\alpha_3 = \cancel{3}$ $/ \cdot 2$

$-2\alpha_1 - \cancel{1} + \alpha_3 = \cancel{-1}$

$5\alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_3 = 0$

$f(2,1,2) = \alpha_1(1,-2,1) + \alpha_2(3,-1,2) + \alpha_3(2,1,2)$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 2 \quad / \cdot(-1)$ $\quad -2$ $\quad -\alpha_2 = 1 \Rightarrow \boxed{\alpha_2 = -1}$

$-2\alpha_1 - \alpha_2 + \alpha_3 = 1$ $\quad -16$ $\quad \alpha_1 + 2\alpha_3 = 1 \quad / \cdot 2$

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 2$ $\quad -1$ $\quad -2\alpha_1 + \alpha_3 = -17$

$5\alpha_3 = -15$

$\Rightarrow \alpha_2 = 0$ $\quad \boxed{\alpha_3 = -3, \ \alpha_1 = 7}$

$\alpha_1 + 2\alpha_3 = 2 \quad / \cdot 2$

$-2\alpha_1 + \alpha_3 = 1$

$5\alpha_3 = 5 \Rightarrow \alpha_3 = 1 \quad \alpha_1 = 0$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 7 & -13 & 7 \\ 6 & -2 & -1 \\ 4 & 1 & -3 \end{pmatrix}$$

$f(8,-6,7) = 1 \cdot e_1 - 1 \cdot e_2 + 1 \cdot e_3 = (33,-16,27) = f(e_1) \overset{?}{=} (1,-1,1)$

$f(-16,7,-13) = -18 \cdot e_1 - 22 \cdot e_2 - 25 \cdot e_3 = (-17,29,-15) = f(e_2) \overset{?}{=} (-18,-22,-25)$

$f(9,-3,7) = 15e_1 + 20e_2 + 22e_3 = (-2,-16,-1) = f(e_3) \overset{?}{=} (15,20,22)$

$f(1,-2,1) = f(1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3) = f(e_1) + f(e_2) + f(e_3) = (14,-3,11)$

$f(3,-1,2) = f(1 \cdot e_1 + 2 \cdot e_2 + 3 \cdot e_3) = f(e_1) + 2f(e_2) + 3 \cdot f(e_3) = (-7,-6,-6)$

$f(2,1,2) = f(-3e_1 - 5e_2 - 6e_3) = -3f(e_1) - 5f(e_2) - 6f(e_3) = (-2,-1,0)$

$$A_f = \begin{pmatrix} 14 & -7 & -2 \\ -3 & -6 & -1 \\ 11 & -6 & 0 \end{pmatrix} = \sum \alpha_i b_i$$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 14$ $\quad \alpha_1 = 1 \quad 14 = 14$

$-2\alpha_1 - \alpha_2 + \alpha_3 = -3$ $\quad \alpha_2 = 3 \quad -2 - 5 + 2 = 3$

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 11$ $\quad \alpha_3 = 2$

$$\mathrm{mat}_{B_2} f = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

57. $f(x)=e^{ax}\cdot P(x)$, $a\in\mathbb{R}$, $a\neq 0$ $\deg P(x)\leq 3$

Dokazati da je operator definisan sa $D(f)=f'$ lin. i naći mat. tog op. u odnosu na bazu:

$B=\{e^{ax}, xe^{ax}, x^2e^{ax}, x^3e^{ax}\}$

$D(\alpha f+\beta g)=(\alpha f+\beta g)'=\alpha f'+\beta g'=\alpha\cdot D(f)+\beta D(g)$

$D(e^{ax})=a\cdot e^{ax}=(a,0,0,0)$

$D(xe^{ax})=e^{ax}+x\cdot ae^{ax}=(1,a,0,0)$

$D(x^2e^{ax})=2xe^{ax}+x^2\cdot ae^{ax}=(0,2,a,0)$

$D(x^3e^{ax})=3x^2e^{ax}+x^3ae^{ax}=(0,0,3,a)$

$$A_D=\begin{pmatrix} a & 1 & 0 & 0\\ 0 & a & 2 & 0\\ 0 & 0 & a & 3\\ 0 & 0 & 0 & a\end{pmatrix}$$

55. $f: V\to V$ V je trodim. v.p.

Da li su:

c) $f(x)=(2x_1+x_2, x_1+x_3, x_3^2)$

d) $f(x)=(x_1-x_2+x_3, x_3, x_2)$

lin. operatori, ako jesu naći njihove mat. u bazi u kojoj su zadane koordinate vektora x i $f(x)$

c) $f(x)=(2x_1+x_2, x_1+x_3, x_3^2)$

$f(\alpha x+\beta y)=f(\underbrace{\alpha x_1+\beta y_1}_{x_1}, \underbrace{\alpha x_2+\beta y_2}_{x_2}, \underbrace{\alpha x_3+\beta y_3}_{x_3})=$

$=(2\alpha x_1+2\beta y_1+\alpha x_2+\beta y_2, \alpha x_1+\beta y_1+\alpha x_3+\beta y_3, \alpha^2x_3^2+2\alpha\beta x_3y_3+\beta^2y_3^2)$

$=(\underline{\alpha(2x_1+x_2)+\beta(2y_1+y_2)}, \underline{\alpha(x_1+x_3)+\beta(y_1+y_3)}, (\alpha x_3+\beta y_3)^2)$

$=\alpha(2x_1+x_2, x_1+x_3,$

nije lin. operator

d) $f(x)=(x_1-x_2+x_3, x_3, x_2)$ $\quad x,y\in W,\ \alpha,\beta\in F$

$f(\alpha x+\beta y)=f(\alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \alpha x_3+\beta y_3)=$

$=(\alpha x_1+\beta y_1-\alpha x_2-\beta y_2+\alpha x_3+\beta y_3, \alpha x_3+\beta y_3, \alpha x_2+\beta y_2)$

$=(\alpha(x_1-x_2+x_3)+\beta(y_1-y_2+y_3), \alpha x_3+\beta y_3, \alpha x_2+\beta y_2)$

$=\alpha(x_1-x_2+x_3, x_3, x_2)+\beta(y_1-y_2+y_3, y_3, y_2)$

$=\alpha f(x)+\beta f(y)$

jeste lin. operator

$B=\{e_1,e_2,e_3\}$

$f(e_1)=(1,0,0)=(1,0,0)$

$f(e_2)=(0,1,0)=(-1,0,1)$

$f(e_3)=(0,0,1)=(1,1,0)$

$$A_f=\begin{pmatrix}1&-1&1\\0&0&1\\0&1&0\end{pmatrix}$$

54. Dokazati da postoji jedinstven lin. o. trodim. v.p. koji prevodi vektore a_1,a_2,a_3 u b_1,b_2,b_3, redom. I naći mat. tog operatora u istoj bazi u kojoj su date i koordinate tih vektora.

b) $a_1=(2,0,3)$, $a_2=(4,1,5)$, $a_3=(3,1,2)$

$b_1=(1,2,-1)$, $b_2=(4,5,-2)$, $b_3=(1,-1,1)$

$f: A\to B$

$\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3=0$ lin. nez. ?

$2\alpha_1+4\alpha_2+3\alpha_3=0$

$\alpha_2+\alpha_3=0$

$3\alpha_1+5\alpha_2+2\alpha_3=0$

$$D=\begin{vmatrix}2&4&3\\0&1&1\\3&5&2\end{vmatrix}\overset{\cdot(-1)}{=}\begin{vmatrix}2&1&3\\0&0&1\\3&3&2\end{vmatrix}=-1\cdot(6-3)=-3\neq0 \Rightarrow \alpha_1=\alpha_2=\alpha_3=0$$

lin. nez. w

$\Rightarrow B=\{a_1,a_2,a_3\}$ su lin. nez. i kako je trodim. v.p. $\Rightarrow B$ je baza

$\Rightarrow \exists!$ lin. o.

Pa prema T: Prostor V je potpuno određen ako su poznate slike elemenata baze; potrebno je odrediti i slike:

$$f(a_1) = f(2,0,3) = f(2\cdot e_1 + 0\cdot e_2 + 3\cdot e_3) = 2f(e_1) + 3f(e_3) = (1,2,-1) = e_1 + 2e_2 - e_3$$

$$f(a_2) = f(4,1,5) = 4f(e_1) + f(e_2) + 5f(e_3) = 4e_1 + 5e_2 - 2e_3$$

$$f(a_3) = f(3,1,2) = 3f(e_1) + f(e_2) + 2f(e_3) = e_1 - e_2 + e_3$$

$$2f(e_1) + 3f(e_3) = e_1 + 2e_2 - e_3$$

$$4f(e_1) + f(e_2) + 5f(e_3) = 4e_1 + 5e_2 - 2e_3$$

$$3f(e_1) + f(e_2) + 2f(e_3) = e_1 - e_2 + e_3 \quad / \cdot (-1)$$

$$2f(e_1) + 3f(e_3) = e_1 + 2e_2 - e_3$$

$$f(e_1) + 3f(e_3) = 3e_1 + 6e_2 - 3e_3 \quad / \cdot (-1)$$

$$f(e_1) = -2e_1 + 4e_2 + 2e_3 = (-2,-4,2) = \frac{1}{3}(-6,-12,6)$$

$$f(e_3) = \frac{e_1 + 2e_2 - e_3 + 4e_1 + 8e_2 + 4e_3}{3} = \frac{5e_1 + 10e_2 - 5e_3}{3} = \frac{1}{3}(5,10,-5)$$

$$f(e_2) = e_1 - e_2 + e_3 + 6e_1 + 12e_2 - 6e_3 + \frac{2}{3}(5e_1 + 10e_2 - 5e_3)$$

$$= \frac{21e_1 + 33e_2 - 15e_3 - 10e_1 - 20e_2 + 10e_3}{3} = \frac{1}{3}(11,13,-5)$$

$$A_f = \frac{1}{3}\begin{pmatrix} -6 & 11 & 5 \\ -12 & 13 & 10 \\ 6 & -5 & -5 \end{pmatrix}$$

53. $f: M_2 \to M_2 \quad f(X) = AX + XB$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

a) Da li je lin. operator?

$$f(\alpha X + \beta Y) = A(\alpha X + \beta Y) + (\alpha X + \beta Y)B$$
$$= A \cdot \alpha X + A \cdot \beta Y + \alpha X \cdot B + \beta Y \cdot B$$
$$= \alpha \cdot (AX) + \beta (AY) + \alpha (X \cdot B) + \beta (YB)$$
$$= \alpha (AX + XB) + \beta (AY + YB)$$
$$= \alpha f(x) + \beta f(y) \Rightarrow \text{jest lin. operator}$$

b) Naći mat. operatora f u odnosu na bazu

$$\mathcal{B} = \left\{ \underset{E_1}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{E_2}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{E_3}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{E_4}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \right\}$$

$$f(E_1) = A \cdot E_1 + E_1 \cdot B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \cdot a + 0 \cdot b & a \cdot 0 + b \cdot 1 \\ c \cdot 0 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g & h \end{bmatrix} = \begin{bmatrix} 0 & b \\ g & d+h \end{bmatrix}$$

$$= 0 \cdot E_4 + b \cdot E_3 + g \cdot E_2 + (d+h) E_1 = (d+h, g, b, 0)$$

$$f(E_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ e & f \end{bmatrix} = \begin{bmatrix} b & 0 \\ d+e & f \end{bmatrix}$$

$$= b \cdot E_4 + 0 \cdot E_3 + (d+e) E_2 + f \cdot E_1 = (f, d+e, 0, b)$$

$$f(E_3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} + \begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} g & a+h \\ 0 & c \end{bmatrix}$$

$$= g \cdot E_4 + (a+h) E_3 + 0 \cdot E_2 + c \cdot E_1 = (c, 0, a+h, g)$$

$$f(E_4) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} e & f \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+e & f \\ c & 0 \end{bmatrix}$$

$$= (a+e) E_4 + f \cdot E_3 + c \cdot E_2 + 0 \cdot E_1 = (0, c, f, a+e)$$

$$A_f = \begin{pmatrix} d+h & f & c & 0 \\ g & d+e & 0 & c \\ b & 0 & a+h & f \\ 0 & b & g & a+e \end{pmatrix}$$

52. $f: M_2 \to M_2$ nad $\mathbb{R}$

$f(X) = A \cdot X \cdot B \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

a) Da li je lin. operator?

$f(\alpha X + \beta Y) = A \cdot (\alpha X + \beta Y) \cdot B = (A \cdot \alpha X + A \cdot \beta Y) \cdot B = A \cdot \alpha X \cdot B + A \cdot \beta Y \cdot B$

$= \alpha \cdot A \cdot X \cdot B + \beta \cdot A \cdot Y \cdot B = \alpha \cdot f(X) + \beta \cdot f(Y)$

jeste lin. operator.

b) Naći mat. operatora u odnosu na bazu:

$$B = \left\{ \underset{E_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{E_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{E_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{E_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

$$f(E_1) = A \cdot E_1 \cdot B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & af \\ ce & cf \end{bmatrix}$$

$$= cf \cdot E_4 + ce \cdot E_3 + af \cdot E_2 + ae \cdot E_1 = (ae, af, ce, cf)$$

$$f(E_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ag & ah \\ cg & ch \end{bmatrix}$$

$$= (ag, ah, cg, ch)$$

$$f(E_3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} be & bf \\ de & df \end{bmatrix}$$

$$= (be, bf, de, df)$$

$$f(E_4) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} bg & bh \\ dg & dh \end{bmatrix}$$

$$= (bg, bh, dg, dh)$$

$$A_f = \begin{bmatrix} ae & ag & be & bg \\ af & ah & bf & bh \\ ce & cg & de & dg \\ cf & ch & df & dh \end{bmatrix}$$

05. Neka je $\mathcal{A}: M_2(\mathbb{R}) \to M_2(\mathbb{R})$; $\mathcal{A}(X) = \frac{1}{2}(X - X^T)$

a) Dokazati da je $\mathcal{A}$ lin. op.

b) Odrediti matricu operatora $\mathcal{A}$ u standardnoj bazi

c) Naći Ker $\mathcal{A}$, Im $\mathcal{A}$, njihove dimenzije i po jednu njihovu bazu

d) Odrediti karakteristični i minimalni polinom matrice operatora.

e) Naći A^n, $n \in \mathbb{N}$

b) $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$A = \text{mat}_B \mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to$ matrica operatora

c) $\text{Ker } \mathcal{A} = \{ X \mid \mathcal{A}(X) = 0 \}$

$= \left\{ X \mid \frac{1}{2}(X - X^T) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ ovo će biti $= 0$ ako su X i X^T simetrične

$= \{ X \mid X = X^T \} = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} =$

$= \left\{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$

$= \mathcal{L}\left(\underbrace{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}}_{} \right)$

3 lin. nezavisna vektora (proveriti lin. komb.

$\sum \alpha_i a_i = 0$

$\alpha_1 = 0$

$\alpha_2 = 0$

$\alpha_2 = 0$

$\alpha_3 = 0$

def $\mathcal{A} = 3$

$\operatorname{Im} \mathcal{A} = \{ \mathcal{A}(X) \mid X \in M_2(\mathbb{R}) \}$

$= \{ \frac{1}{2}(X - X^T) \mid X \in M_2(\mathbb{R}) \}$

$= \{ \frac{1}{2} \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix} \mid b, c \in \mathbb{R} \} = \{ (b-c) \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \mid (b-c) \in \mathbb{R} \}$

$= \mathcal{L} \left(\left\{ \begin{matrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{matrix} \right\} \right)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$

rang $\mathcal{A} = 1 = \dim \operatorname{Im} \mathcal{A}$ $3 + 1 = 4$

d) Karakteristični polinom matrice (mat. mora biti kvadratna)

$$(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

T. Svaka kvadratna matrica anulira svoj karakteristični polinom (Keli-Hamiltonov teorem)

Minimalni polinom mat. je minimalni faktor karakterističnog polinoma kojeg anulira ta matrica.

$$P_A(\lambda) = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & \frac{1}{2} - \lambda & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} - \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = (-\lambda) \cdot \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda)\left(\left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} \right) = \lambda^2 \cdot \left(\frac{1}{2} - \lambda - \frac{1}{2}\right)\left(\frac{1}{2} - \lambda + \frac{1}{2}\right) =$$

$= \lambda^3 (\lambda - 1) \rightarrow$ karakteristični pol. operatora tj. njegova matrice

Prema K.H. teoremu:

$$A^3 \cdot (A - I) = 0 \implies A^4 = A^3$$

Minimalni pol. ima iste nule kao i karakteristični pol.; pa su kandidati za nule:

$\lambda(\lambda-1),\ \lambda^2(\lambda-1),\ \lambda^3(\lambda-1)$

treba provjeriti da li je:

$$A\cdot(A-I)\overset{?}{=}0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ minimalni pol. je $\lambda(\lambda-1)$

e) $A\cdot(A-I)=0$

$A^2=A$ (*)

$A^3=A^2\cdot A=A\cdot A=A^2=A$

$A^4=\dots=A$

$\vdots$

$A^m=A$

Mat. indukcijom: $P: A^m=A$

$T: A^{m+1}=A$

$A^{m+1}=A^mA=A\cdot A=A^2=A$